International Journal of Theoretical Physics, Vol. 28, No. 10, 1989

Reductions and Quantization

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Received April 12, 1989

The interchangeability of the Marsden–Weinstein reduction procedure and the Kostant–Souriau geometric quantization is studied by detailed examination of a concrete dynamical system—the so-called MIC-Kepler problem. It is proved that some stages of reduction plus geometric quantization technique produce the complete quantum spectrum of the system, while others give part of it or nothing.

1. INTRODUCTION

In a sense the standard description of the classical dynamical system (P, ω, H) in terms of a symplectic manifold (P, ω) and a smooth function H on P (the Hamiltonian of the system) seems to be ideally suited for the transition to quantum mechanics. I have in mind the Kostant–Souriau geometric quantization scheme, which aims at an extension of the Schrödinger quantization procedure to arbitrary symplectic manifolds. The objects involved in ordinary quantization get in this approach a geometrical interpretation and reflect the topology of the phase space. The starting point is the observation that the symplectic form ω induces a Poisson bracket operation under which the smooth functions $C^\infty(P)$ form a Lie algebra. The problem of constructing representations of this algebra was first raised by Dirac. When further a Lie group G acts symplectically (canonically) on the symplectic (phase space) manifold leaving the Hamiltonian H invariant, one says that we have a Hamiltonian system with symmetry (P, ω, H, G). Moreover, when the action of G is such that (P, ω) is a Hamiltonian G-space, the representations of $C^\infty(P)$ provide a means of constructing representations of the Lie group G. If (P, ω) is viewed as the phase space of the classical Hamiltonian system with symmetry (P, ω, H, G), they arise

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in the process of quantizing this system. Classically the symmetries result in the appearance of constraints. Factoring out the symmetries, one gets the so-called reduced phase space (P_μ, ω_μ) and the reduced dynamics on it described by the Marsden and Weinstein (1974) theorem (see Section 2 for details). Thus, each Hamiltonian system with symmetry has two symplectic faces, the initial and reduced phase spaces (P, ω) and (P_μ, ω_μ), respectively. Of great importance is the absence of a formal distinction between working on (P, ω) or (P_μ, ω_μ) at the classical level. But these two representations of one and the same mechanical system are not necessarily equivalent on the quantum level. The problem to correlate properly the quantization of extended and reduced phase space has still to be settled in general. For example, the geometric quantization of the extended and reduced phase spaces has been proved (Puta, 1984) to be equivalent within the cotangent bundle category where the starting and reduced phase spaces are cotangent bundles provided with their canonical symplectic structures. At the other extreme is the case when the symplectic manifold to be reduced is a compact Kaehler manifold (Guillemin and Sternberg, 1982). The real situation in mechanics is somewhere between, as when one starts with a cotangent bundle and then, after reduction, obtains either a compact Kaehler manifold or a cotangent bundle whose symplectic form is not the canonical one.

Substantial progress toward clarifying the situation has already been made (Puta, 1984; Guillemin and Sternberg, 1982; Gotay, 1986; Blau, 1988), but one of the problems is the lack of examples in which the corresponding physical systems are well understood. The purpose of this paper is to provide such a description in the case of the MIC-Kepler problem. Studying its symplectic faces $(T^*\dot{R}^4, \Omega)$, $(T^*\dot{R}^3, \Omega_\mu)$, $(P^1 \times P^1, \Omega_\mu(E))$, and $(T^*R^+, dp_r \wedge dr)$, we find that geometric quantization fails to recognize them as unique.

2. PRELIMINARIES

In this section I collect the exact statements of reduction theorems and give a brief outline of geometric quantization program in the form needed later. Here I fix also notation and conventions.

Theorem 1 (Marsden and Weinstein, 1974). Let (P, ω) be a symplectic manifold on which a Lie group G acts symplectically and $J\colon P \to \mathcal{G}^*$ (the dual of the Lie algebra $\mathcal{G}$ of G) be an Ad^*-equivariant moment map. Assume that $\mu \in \mathcal{G}^*$ is a regular value of J and that the isotropy subgroup G_μ acts freely and properly on $J^{-1}(\mu)$. Then $P_\mu = J^{-1}(\mu)/G_\mu$ is a symplectic manifold with the symplectic form ω_μ determined by $\pi_\mu^*\omega_\mu = i_\mu^*\omega$, where $\pi_\mu\colon J^{-1}(\mu) \to P_\mu$ is the canonical projection and $i_\mu\colon J^{-1}(\mu) \to P$ is the inclusion map. Let $H: P \to R$ be G-invariant. A Hamiltonian flow on P_μ is induced whose Hamiltonian H_μ satisfies $H_\mu \circ \pi_\mu = H \cdot i_\mu$.

Remark 1. If the Hamiltonian system (P, ω, H) admits a symmetry group which commutes with G, then $(P_\mu, \omega_\mu, H_\mu)$ preserves this symmetry.

Theorem 2 (Kummer, 1981). Let P be a cotangent bundle T^*M and G a one-parameter Lie group acting freely and properly on M. Let $M \to N = M/G$ be the induced principal fiber bundle and $\tilde{\alpha}$ *a* connection one-form on it. The reduced manifold P_μ is symplectomorphic to T^*N endowed with a symplectic form given by the canonical one plus a "magnetic term" $\mu\tau_N^* d\tilde{\alpha}$ (where τ_N is the canonical projection τ_N: $T^*N \to N$).

A thorough discussion concerning the reduction of symplectic manifolds and detailed examination of the classical examples from the modern point of view can be found in Abraham and Marsden (1978), Marmo *et al.* (1985), and Libermann and Marle (1987).

The geometric quantization scheme of Kostant (1970) and Souriau (1970) associates to any quantizable phase space (X, Ω) a Hilbert space $\mathscr{H}$ and to a subalgebra of the smooth functions on X quantum operators on $\mathscr{H}$. One says that the symplectic manifold is quantizable if $[\Omega/2\pi]$ is in the image of the map

$$H^2_{\text{Chech}}(X, \mathbb{Z}) \to H^2_{\text{de Rham}}(X)$$

where $[\cdot]$ denotes the de Rham cohomology class. When X is a compact this (pre)quantum condition on the form Ω amounts to

$$\frac{1}{2\pi}\int_\sigma \Omega \in \mathbb{Z} \qquad \text{for all} \quad \sigma_2 \in H_2(X, \mathbb{Z}) \tag{1}$$

If (X, Ω) is quantizable, there exists a (prequantum) line bundle $L \to X$ whose Chern class is $[\Omega/2\pi]$ equipped with a connection ∇ whose curvature form is $-i\Omega$ and Hermitian inner product $h(\cdot, \cdot)$ which is invariant under parallel transport. The Hilbert space $\mathscr{H}$ is built by the polarized section of the quantum line bundle $Q \to X$, where $Q = L \otimes N_F^{1/2}$ and $N_F^{1/2}$ is the line bundle of half-forms normal to the polarization F (which is supposed to be invariant $[X_f, F] \subset F$).

If $\psi = s \otimes \nu$, where $s \in \Gamma(L)$, $\nu \in \Gamma(N_F^{1/2})$ and $\psi \in \Gamma(Q)$ are sections of the respective line bundles, then one associates to the classical observable f a quantum operator $\hat{f}$ acting on $\mathscr{H}$ by

$$\hat{f}(\psi) = [-iX_f - \theta(X_f) + f]s \otimes \nu - is \otimes \mathscr{L}(X_f)\nu \tag{2}$$

Here X_f is the Hamiltonian vector field generated by $f(i(X_f)\Omega = -df)$, θ is the potential one-form of $\Omega(d\theta = \Omega)$, and $\mathscr{L}(X_f)$ is the Lie derivative with respect to X_f. Complete exposition of geometric quantization can be found in Simms and Woodhouse (1976), Sniatycki (1980), and Tuynman (1985).

3. THE MIC-KEPLER PROBLEM

The MIC-Kepler problem (Iwai and Uwano, 1986; see also Mladenov and Tsanov, 1987) is the Hamiltonian system

$$(T^*\dot{R}^3, \Omega_\mu, H_\mu) \tag{3}$$

where

$$T^*\dot{R}^3 = \{(q, p) \in R^3 \times R^3;\ q \neq 0\}$$

$$\Omega_\mu = d\theta + \sigma_\mu, \qquad \theta = \sum p_j\, dq_j, \qquad \sigma_\mu = -\mu/(2|q|^3)\varepsilon_{ijk} q_i\, dq_j \wedge dq_k \tag{4}$$

$$H_\mu = \tfrac{1}{2}|p|^2 - \alpha/r + \mu^2/2r^2, \qquad |q|^2 = q_1^2 + q_2^2 + q_3^2 = r^2, \qquad \alpha, \mu \in R, \qquad \alpha > 0$$

This Hamiltonian system describes the motion of a charged particle in the presence of a Dirac monopole field $B_\mu = -\mu q/r^3$, a Newtonian potential $-\alpha/r$, and a centrifugal potential $\mu^2/2r^2$. Further, I refer to this system as the MIC-Kepler problem, as McIntosh and Cisneros studied it first using a vector potential. It turns out that for $E < 0$, the energy level submanifolds $H_\mu^{-1}(E)$ consist only of closed orbits. This implies the presence of "hidden" symmetry and "accidental" degeneracy of the energy spectrum. Actually, this "hidden" $SO(4)$ symmetry of the Hamiltonian system $(T^*\dot{R}^3, \Omega_\mu, H_\mu)$ is generated by constants of motion

$$\mathbf{L}^\mu = \mathbf{q} \times \mathbf{p} + \mu\mathbf{q}/r, \qquad \mathbf{A}^\mu = (\mathbf{L}^\mu \times \mathbf{p} + \alpha\mathbf{q}/r)/(-2H_\mu)^{1/2}$$

which have an interpretation as total momentum and generalized Runge-Lenz vector. All this resembles the ordinary Kepler problem and in fact it can be viewed as a limit of the one-parameter family of deformations (Bates, 1988). The standard Kepler problem ($\mu = 0$) has been quantized geometrically by Simms (1973) and Mladenov and Tsanov (1985) in higher dimensions.

Here, I apply the geometric quantization to extended and reduced phase spaces of the Hamiltonian system (4); which leads (surprisingly enough) to the same result which is the statement of the following theorem.

Theorem 3. The discrete spectrum (bound states) of the MIC-Kepler problem (3) (α and μ fixed) consists of the energy levels

$$E_N = -\alpha^2/2N^2, \qquad N = |\mu| + 1, |\mu| + 2, \ldots \tag{5}$$

with multiplicities

$$m(E_N) = N^2 - \mu^2 \tag{6}$$

Theorem 3 will be proved in Sections 5 and 6, dealing with extended and reduced phase spaces, respectively.

Remark 2. Prequantization of $(T^*\dot{R}^3, \Omega_\mu)$ selecting the integral symplectic forms produces immediately the magnetic charge quantization $\mu = 0, \pm\frac{1}{2}, \pm 1, \ldots$ (cf. Section 6). Unfortunately, the geometric quantization scheme as described in Section 2 cannot be applied, because there is no way to dispense with the use of an invariant polarization.

Remark 3. Extensions viewed as nonbijective transformations have been applied with great success by Kibler and Negadi (1984) to various problems in physics and chemistry (see also Davtyan *et al.*, 1987).

Remark 4. Gotay and Tuynman (1988) have proved that all symplectic manifolds can be obtained by means of a symplectic reduction of appropriate $(R^{2n}, \omega_{\text{can}})$.

4. THE CONFORMAL KEPLER PROBLEM AND ITS REDUCTION

Let us start with the symplectic manifold

$$T^*\dot{R}^4 = \{(x, y) \in R^4 \times R^4, x \neq 0\} \tag{7}$$

with the standard symplectic form

$$\Omega = dy \wedge dx = \sum_{j=1}^{4} dy_j \wedge dx_j \tag{8}$$

Next, introduce three Hamiltonian functions on the phase space $(T\dot{R}^4, \Omega)$. First, the Hamiltonian of the conformal Kepler problem

$$H = (|y|^2 - 8\alpha)/8|x|^2, \qquad \alpha \text{ a fixed positive constant} \tag{9}$$

Second, the Hamiltonian of a harmonic oscillator

$$K = (|y|^2 + \lambda^2|x|^2)/2, \qquad \lambda \text{ an arbitrary positive constant} \tag{10}$$

Third, a momentum Hamiltonian

$$M = \tfrac{1}{2}(x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3) \tag{11}$$

Obviously, we have

$$4|x|^2(H + \lambda^2/8) = K - 4\alpha \qquad 12)$$

which means that the energy hypersurfaces $H = E = -\lambda^2/8$ and $K = 4\alpha$ coincide. Moreover, the flows defined by the Hamiltonians H and K on the level sets

$$H^{-1}(E) \equiv K^{-1}(4\alpha) \tag{13}$$

coincide up to a monotonic change of parameter, as there the corresponding Hamiltonian vector fields X_H and X_K satisfy

$$4|x|^2 X_H = X_K \tag{14}$$

For an arbitrary choice of the positive constant λ, introduce the complex coordinates

$$z_1 = \lambda(x_1 + ix_2) - i(y_1 + iy_2), \qquad z_2 = \lambda(x_3 + ix_4) - i(y_3 + iy_4)$$

$$z_3 = \lambda(x_1 - ix_2) - i(y_1 - iy_2), \qquad z_4 = \lambda(x_3 - ix_4) - i(y_3 - iy_4)$$

In these coordinates $T^*\dot{R}^4 = \mathbb{C}^4 \backslash D$, where

$$D = \{z \in \mathbb{C}^4, \, z_1 = -\bar{z}_3, \, z_2 = -\bar{z}_4\} \tag{15}$$

and the symplectic form Ω is a multiple of the standard Kaehler form on $\mathbb{C}^4$,

$$\Omega = \frac{i}{4\lambda} dz \wedge d\bar{z} = \frac{i}{4\lambda} \sum_{j=1}^{4} dz_j \wedge d\bar{z}_j \tag{16}$$

the Hamiltonian functions K and M can also be easily expressed in these coordinates as

$$K = (|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2)/4 \tag{17}$$

and

$$M = (|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2)/8\lambda \tag{18}$$

The Hamiltonians K and M as well as the symplectic form Ω are well defined on the manifold,

$$\dot{\mathbb{C}}^4 = \mathbb{C}^4 \backslash \{0\} \supset T^*\dot{R}^4 \tag{19}$$

Denote by K_t and M_s the flows of the Hamiltonian systems $(\dot{\mathbb{C}}^4, \Omega, K)$ and $(\dot{\mathbb{C}}^4, \Omega, M)$.

Lemma 1. For any $z \in \dot{\mathbb{C}}^4$ and $s, t \in R$, we have

$$K_t z = (e^{i\lambda t} z_1, \, e^{i\lambda t} z_2, \, e^{i\lambda t} z_3, \, e^{i\lambda t} z_4) \tag{20}$$

$$M_s z = (e^{is/2} z_1, \, e^{is/2} z_2, \, e^{-is/2} z_3, \, e^{-is/2} z_4) \tag{21}$$

In particular, the flows of all three Hamiltonians H, K, and M commute where defined.

By Lemma 1 the flow M_s defines a symplectic action of the circle group $U(1)$ on the manifold $\dot{C}^4$. The moment map for this action is M itself. Note that the set D defined in (15) is invariant with respect to this $U(1)$ action. Thus, $T^*\dot{R}^4$ is also invariant, as well as the Hamiltonian H, and we may apply Theorem 2 to reduce the Hamiltonian system $(T^*\dot{R}^4, \Omega, H)$ with respect to the $U(1)$ action (21). The result is the following proposition, established by Iwai and Uwano (1986) (see also Mladenov and Tsanov, 1987).

Proposition. Let $\mu \in R$. Then

$$M^{-1}(\mu)/U(1) \cong T^*\dot{R}^3 \tag{22}$$

and the reduction of the form Ω and the Hamiltonian H give Ω_μ and H_μ, i.e., the result of the reduction is the MIC-Kepler problem (3).

Accordingly, constants of motion for the conformal Kepler problem

$$M_1 = (z_1\bar{z}_2 + z_2\bar{z}_1 - z_3\bar{z}_4 - z_4\bar{z}_3)/8\lambda$$

$$M_2 = (z_1\bar{z}_2 - z_2\bar{z}_1 + z_3\bar{z}_4 - z_4\bar{z}_3)/8\lambda i$$

$$M_3 = (|z_1|^2 - |z_2|^2 - |z_3|^2 + |z_4|^2)/8\lambda$$

$$A_1 = (z_1\bar{z}_2 + z_2\bar{z}_1 + z_3\bar{z}_4 + z_4\bar{z}_3)/8\lambda$$

$$A_2 = (z_1\bar{z}_2 - z_2\bar{z}_1 - z_3\bar{z}_4 + z_4\bar{z}_3)/8\lambda i$$

$$A_3 = (|z_1|^2 - |z_2|^2 + |z_3|^2 - |z_4|^2)/8\lambda$$

fall under reduction as constants of motion $\mathbf{L}^\mu$, $\mathbf{A}^\mu$ of the MIC-Kepler problem.

5. QUANTIZATION OF THE EXTENDED PHASE SPACE

Under the reduction the energy-momentum manifold

$$\mathcal{M}(\lambda, \mu) = \{(x, y) \in T^*\dot{R}^4;\ K = 4\alpha,\ M = \mu\} \tag{23}$$

is mapped by π_μ onto the energy hypersurface $H_\mu = -\lambda^2/8$ $[\lambda = (-8E)^{1/2}]$ of the MIC-Kepler problem. In order that $\mathcal{M}(\lambda, \mu)$ be nonempty, λ and μ must obey

$$\lambda|\mu| \le 2\alpha \tag{24}$$

In this section it is assumed that $\lambda|\mu| < 2\alpha$ holds strongly. Comments on this point can be found in Section 6. Now let us change the point of view and look at $(T^*\dot{R}^4, \Omega)$ as an extension of $(T^*\dot{R}^3, \Omega_\mu)$. Working with complex coordinates, choose the polarization F to be spanned by the antiholomorphic directions $\{\partial/\partial\bar{z}_1, \partial/\partial\bar{z}_2, \partial/\partial\bar{z}_3, \partial/\partial\bar{z}_4\}$ and adapted to it potential one-form $\theta = -(i/4\lambda)\bar{z}\,dz$. The Hilbert space $\mathcal{H}$ associated with the phase space $(T^*\dot{R}^4, \Omega)$ consists of "wave functions" ψ in the form $\psi = \varphi \otimes \nu$, where φ is a holomorphic function and $\nu = (dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4)^{1/2}$.

Essentially, the idea of Dirac's method of quantization in the presence of constraints in the phase space is that they must be enforced quantum mechanically if they have not been eliminated classically. Since the constraints specifying the energy-momentum manifold $\mathcal{M}(\lambda, \mu)$ are given by $K = 4\alpha$, $M = \mu$, it follows that the physically admissible quantum states are those which belong to the subspace $\mathcal{H}_\mu$ of $\mathcal{H}$ defined by

$$\mathcal{H}_\mu = \{\psi \in \mathcal{H};\ \hat{K}\psi = 4\alpha\psi,\ \hat{M}\psi = \mu\psi\}$$

Now, we have

$$\hat{K}\psi = \lambda\left(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3} + z_4\frac{\partial}{\partial z_4} + 2\right)\varphi \otimes \nu$$

$$= \lambda(\mathcal{N}+2)\psi = 4\alpha\psi, \qquad \mathcal{N} = 0, 1, 2, \ldots$$

$$\hat{M}\psi = \frac{1}{2}\left(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} - z_3\frac{\partial}{\partial z_3} - z_4\frac{\partial}{\partial z_4}\right)\varphi \otimes \nu = \mu\psi$$

where φ is a homogeneous polynomial of degree $\mathcal{N} \geq 0$ in z's. Introducing $N = \mathcal{N}/2 + 1$ and solving

$$2N(-8E)^{1/2} = 4\alpha$$

one gets (5) as well

$$n_1 + n_2 + n_3 + n_4 = 2N - 2 \tag{25}$$

$$n_1 + n_2 - n_3 - n_4 = 2\mu, \qquad n_i \geq 0, \quad i = 1, 2, 3, 4 \tag{26}$$

which is equivalent to

$$n_1 + n_2 = N + \mu - 1 = \mathcal{N}_1 \tag{27}$$

$$n_3 + n_4 = N - \mu - 1 = \mathcal{N}_2 \tag{28}$$

By (26) the magnetic charge is quantized according to Dirac's prescription

$$\mu = 0, \pm\tfrac{1}{2}, \pm 1, \pm\tfrac{3}{2}, \ldots \tag{29}$$

Combined, (27) and (28) ensure that $N = |\mu| + 1, |\mu| + 2, \ldots$. To find the multiplicities $m(E_N)$, remark that φ reduces to a product $\varphi_1(z_1, z_2)\varphi_2(z_3, z_4)$ of homogeneous polynomials in two variables of degree $\mathcal{N}_1, \mathcal{N}_2$, respectively. The dimensionality of the Hilbert space $\mathcal{H}_{\mu,N}$ is then

$$m(E_N) = (\mathcal{N}_1 + 1)(\mathcal{N}_2 + 1) = N^2 - \mu^2 \tag{30}$$

Remark 5. The Hilbert space $\mathcal{H}_{\mu,N}$ carries $(\mathcal{N}_1/2, \mathcal{N}_2/2)$ unitary irreducible representations of $SU(2) \times SU(2)$ (μ half-integer) or $SO(4)$ (μ integer). The wave functions within $\mathcal{H} = \bigoplus \mathcal{H}_{\mu,N}$ are labeled by four quantum numbers which are eigenvalues of a complete set of commuting operators: $\hat{M}(\mu)$, $\hat{H}(N)$, $\hat{M}_3(m)$, and $\hat{A}_3(n)$, where

$$\hat{M}_3\psi = \frac{1}{2}\left(z_1\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_2} - z_3\frac{\partial}{\partial z_3} + z_4\frac{\partial}{\partial z_4}\right)\psi = m\psi$$

$$\hat{A}_3\psi = \frac{1}{2}\left(z_1\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3} - z_4\frac{\partial}{\partial z_4}\right)\psi = n\psi$$

6. QUANTIZATION OF THE ORBIT MANIFOLD

As mentioned in Section 3, the energy level submanifolds $H_\mu^{-1}(E)$ consist only of closed orbits for $E<0$. Thus, one can factorize $H_\mu^{-1}(E)$ by the dynamical flow and the so-obtained manifold $H_\mu^{-1}(E)/U(1)$ is called the orbit manifold $\mathcal{O}_\mu(E)$. Its structure is described by the following theorem.

Theorem 4 (Mladenov and Tsanov, 1987). Let $E<0$ and define $\lambda = (-8E)^{1/2}$. Then:

(i) If $\lambda|\mu|<2\alpha$, then $\mathcal{O}_\mu(E)\cong P^1\times P^1$.
(ii) If $\lambda|\mu|=2\alpha$, then $\mathcal{O}_\mu(E)\cong P^1$.
(iii) If $\lambda|\mu|>2\alpha$, then $H_\mu^{-1}(E)=\phi$.

Moreover, the reduced symplectic form on $\mathcal{O}_\mu(E)$ is

$$\Omega_\mu(E) = \frac{2\pi(2\alpha+\lambda\mu)}{\lambda}\,\omega_1 + \frac{2\pi(2\alpha-\lambda\mu)}{\lambda}\,\omega_2 \tag{31}$$

where

$$\omega_j = \frac{i}{2\pi}\,\frac{d\mathbf{z}_j\wedge d\bar{\mathbf{z}}_j}{(1+|\mathbf{z}_j|^2)^2}, \qquad j=1,2 \tag{32}$$

for any pair of nonhomogeneous coordinates $(\mathbf{z}_1, \mathbf{z}_2)$ on $P^1\times P^1$.

The above theorem reduces the quantization of the MIC-Kepler problem to the geometric quantization of the compact Kaehler manifold $P^1\times P^1(P^1)$. Applying the geometric quantization scheme to the orbit manifold $\mathcal{O}_\mu(E)$ amounts in quantum mechanical terms to the transition from the Schrödinger to the Heisenberg picture, and, as we shall see seen, leads also to Theorem 3. The proof of Theorem 4 will not be reproduced here, as it can be found in Mladenov and Tsanov (1987). It is based on Lemma 2 below and that is why I proceed to its formulation. By (20) and (21) the flows K_t, M_s define a Hamiltonian action of the torus $U(1)\times U(1)$ on $\dot{\mathbb{C}}^4$. Denote by

$$J:\quad \dot{\mathbb{C}}^4 \to u^*(1)\times u^*(1)\cong R^2 \tag{33}$$

the associated momentum map of this action. Explicitly, we have

$$J(z)=(K(z), M(z))$$

This is all we need for statement of the following lemma.

Lemma 2. $\mathcal{O}_\mu(E)\cong J^{-1}(4\alpha, \mu)/U(1)\times U(1)$.

Using (17) and (18), we see that the system $K=4\alpha$, $M=\mu$, is equivalent to

$$|z_1|^2+|z_2|^2=4(2\alpha+\lambda\mu), \qquad |z_3|^2+|z_4|^2=4(2\alpha-\lambda\mu) \tag{34}$$

whence

$$J^{-1}(4\alpha, \mu)=\begin{cases} S^3\times S^3 & \text{when } \lambda|\mu|<2\alpha \\ S^3 & \text{when } \lambda|\mu|=2\alpha \\ \phi & \text{when } \lambda|\mu|>2\alpha \end{cases} \tag{35}$$

Define a projection (two Hopf maps)

$$p: \quad S^3\times S^3\to P^1\times P^1$$

by

$$p(z_1, z_2, z_3, z_4)=((z_1, z_2), (z_3, z_4)) \tag{36}$$

where (z_1, z_2) and (z_3, z_4) are homogeneous coordinates on $P^1\times P^1$. The reduced symplectic form $\Omega_\mu(E)$ is computed by the definition given in Theorem 1,

$$p^*\Omega_\mu(E)=\Omega|_{S^3\times S^3}$$

The Kaehler forms ω_1, ω_2 defined in (32) generate $H^2(P^1\times P^1, \mathbb{Z})$ (cf. Griffiths and Harris, 1978)

$$H^2(P^1\times P^1, \mathbb{Z})=\mathbb{Z}\oplus\mathbb{Z} \tag{37}$$

and

$$c_1(N_F^{1/2})=-\tfrac{1}{2}c_1(P^1\times P^1)=-([\omega_1]+[\omega_2]) \tag{38}$$

The existence of prequantum line bundle L is obviously a condition on E, which determines the energy spectrum.

We combine (1), (31), and (37) to obtain

$$\frac{1}{2\pi}\Omega_\mu(E)=N_1\omega_1+N_2\omega_2 \tag{39}$$

for some integers $N_1, N_2\geq 1$.

So we have that

$$2\alpha + \lambda\mu = \lambda N_1$$
$$2\alpha - \lambda\mu = \lambda N_2$$

whence

$$\mu = \tfrac{1}{2}(N_1 - N_2), \qquad \lambda = 4\alpha/(N_1 + N_2) \tag{40}$$

Introducing $N = \frac{1}{2}(N_1 + N_2)$, we obtain $N_1 = N + \mu$, $N_2 = N - \mu$, where $N \geq |\mu| + 1$, and finally the energy spectrum (5) of the MIC-Kepler problem. The multiplicities of these levels $m(E_N)$ coincide with the dimensions of the spaces of holomorphic sections of the line bundles Q over $P^1 \times P^1$. If $Q_N \to P^1 \times P^1$ is such a quantum bundle with

$$c_1(Q_N) = (N_1 - 1)[\omega_1] + (N_2 - 1)[\omega_2]$$

then by the Riemann-Roch-Hirzebruch theorem for compact complex surfaces (Hirzebruch, 1966) and the Kodaira vanishing theorem (Griffiths and Harris, 1978) we have

$$m(E_N) = \dim H^0(P^1 \times P^1, Q_N) = N_1 N_2 = N^2 - \mu^2$$

Remark 6. Multiplicities can be found also by the methods of enumerative geometry as done in Gaeta and Spera (1988) in the case $N_1 = N_2$.

Members of the complete set of observables M_3 and A_3 survived under reduction to $\mathcal{O}_\mu(E_N)$ can be expressed in nonhomogeneous coordinates $(\mathbf{z}_1, \mathbf{z}_2) = (z_2/z_1, z_4/z_3)$ on $P^1 \times P^1$ as follows:

$$M_3^{\mu,N} = \frac{N_1}{2}\frac{1 - |\mathbf{z}_1|^2}{1 + |\mathbf{z}_1|^2} - \frac{N_2}{2}\frac{1 - |\mathbf{z}_2|^2}{1 + |\mathbf{z}_2|^2}$$

$$A_3^{\mu,N} = \frac{N_1}{2}\frac{1 - |\mathbf{z}_1|^2}{1 + |\mathbf{z}_1|^2} + \frac{N_2}{2}\frac{1 - |\mathbf{z}_2|^2}{1 + |\mathbf{z}_2|^2}$$

The corresponding quantum operators $\hat{M}_3^{\mu,N}$ and $\hat{A}_3^{\mu,N}$ act on the sections of the quantum line bundle $Q_N \cong L_1^{N_1} \otimes L_2^{N_2}$. Here L_i $(i = 1, 2)$ stand for the dual of the universal line bundle over the respective P^1 factor in $\mathcal{O}_\mu(E_N)$ and $\mathcal{N}_i$ is its tensor power. The holomorphic sections $\Psi_i \in \Gamma(L_i)$ of these line bundles span the carrier spaces of spin $s_i = N_i/2$ representations of $SU(2)$ (cf. Remark 5).

It is (at least mathematically) sensible to prequantize also the orbit manifold when $\lambda|\mu| = 2\alpha$. Classically this is the energy level of the system which corresponds to the minimal value $-\alpha^2/\mu^2$ of the potential $U_\mu(r) = \mu^2/2r^2 - \alpha/r$.

Curiously, the procedure gives once more the quantization of the magnetic charge obtained independently by the above argument [see (40) and Section 5].

Remark 7. The MIC-Kepler problem offers also the possibility to regularize the standard Kepler problem in a different way from that proposed in Moser (1970), Kummer (1982), Cordani (1986), and Vivarelli (1986). More details can be found in Mladenov and Tsanov (1987).

Remark 8. It turns out that the MIC-Kepler problem and the Taub-NUT problem with negative mass parameter are "hiddenly" symplectomorphic systems with identical degeneracies (Cordani *et al.*, 1988).

Remark 9. Reducing $(T^*\dot{R}^3, \Omega_\mu, H_\mu)$ with respect to the obvious $SO(3)$ Hamiltonian actions, one gets $(T^*R^+, dp_r \wedge dr, p_r^2/2 + l^2/2r^2 - \alpha/r)$ but in this case geometric quantization does not provide any information about the spectrum.

Beyond doubt, the reduction-quantization relationship is a challenge for the geometric quantization program and deserves further exploration in order to be made into a strong theory.

ACKNOWLEDGMENTS

I thank Prof. Abdus Salam, the International Atomic Energy Agency, and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste. I also express my gratitude to Dr. V. Tsanov for invaluable discussions and frequent collaboration. Mladenov and Tsanov (1987) is a close companion to the work described here.

This work was partially supported by contracts 330/88 and 911/88 of the Ministry of Culture, Science and Education.

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